MTH 201
Multivariable calculus and differential equations
Homework 3
Functions of several variables
Limit, continuity, and partial derivatives

1. Show that
   (a) The set $\{(x, y) \in \mathbb{R}^2 : y = x^2\}$ is a closed subset of $\mathbb{R}^2$.
   (b) The set $\{(x, y) \in \mathbb{R}^2 : x > 0\}$ is an open subset of $\mathbb{R}^2$.
   (c) The set $\{(x, y, z) \in \mathbb{R}^3 : 3x + 2y + z = 6\}$ is a closed subset of $\mathbb{R}^3$. (HW)

2. Describe the domain for each of the following functions
   (a) $f(x, y) = \frac{\sqrt{x+y+1}}{x-1}$ (b) $f(x, y) = x \log(y^2 - x)$
   (c) $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2-1}}$ (d) $f(x, y, z) = xy \sin z + \log(z - y)$.

3. Describe the level curve for each of the following functions
   (a) $f(x, y) = y^2 - x^2$ (HW)
   (b) $f(x, y) = 4x^2 + y^2$.

4. Consider the function defined by $f(x, y) = \frac{2^{99}x^2y^2}{x^2+y^2}$, $(x, y) \neq (0, 0)$. Use $\epsilon - \delta$ definition of limit to show that (HW)
$$\lim_{(x,y)\to(0,0)} f(x, y) = 0.$$

5. For each of the following functions, find $\lim_{x\to 0}\left(\lim_{y\to 0} f(x, y)\right)$, $\lim_{y\to 0}\left(\lim_{x\to 0} f(x, y)\right)$, and $\lim_{(x,y)\to(0,0)} f(x, y)$, if exist
   (a) $f(x, y) = \frac{x^2-y^2}{x^2+y^2}$
   (b) $f(x, y) = \frac{x^2+y^2}{x^2+2y^2}$
   (c) $f(x, y) = \frac{x^2y^2}{x^4+y^2}$
   (d) $f(x, y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$
   (e) $f(x, y) = \frac{x \sin(1/x)+y}{x+y}$ (HW)
   (f) $f(x, y) = \frac{x^2 \sin^2 y+y^2 \sin^2 x}{x^2+y^2}$

6. Which of the functions in Question 5 can be made continuous at $(0, 0)$ by suitably defining them at $(0, 0)$.

7. Show that the function f defined by $f(X) = |X|$, $X = (x, y, z) \in \mathbb{R}^3$, is continuous on $\mathbb{R}^3$.

8. Let $Y \in \mathbb{R}^3$ be a point. Show that the function f defined by $f(X) = Y \cdot X$, $X = (x, y, z) \in \mathbb{R}^3$, is continuous on $\mathbb{R}^3$. (HW)

9. Consider the function defined by $f(x, y) = 0$, $xy \neq 0$ and $f(x, y) = 1$, $xy = 0$. Find partial derivatives if they exist. (HW)